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On Singular Phenomena in Certain Time-Optimal Problem

WLADYSLAW HEJMO
Institute of Control Engineering, Cracow University of Technology, ul. Warszawska 24, 31-155 Krakow, Poland (e-mail: hejmo@kts.edison.pk.edu.pl)

JACENTY KLOCH
Mathematical Institute of Polish Academy of Sciences, ul.Sw.Tomasza 30, 31-027 Krakow, Poland

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Abstract. The purpose of this paper is to present the solution of time-optimal problem of the controlled object the dynamics of which is given by: $\dot{x} = y, \dot{y} = f(x) + u$, where $|u| \leqslant 1$ and motion resistance function $f(x) = 0$ if $x \leqslant 0$, $f(x) = -A$ if $x > 0$ where $0 \leqslant A < 1$. That model describes dynamics of a very important class of industrial installations. As the time-optimal problem will be understood a transfer of the initial state $\mathbf{z}_0 = (x_0, y_0) \in R^2$ to the target state $\mathbf{z}_1 = (x_1, 0), x_1 \geqslant 0$ in a minimum time $t^* < \infty$. There has been shown that in the formula defining resistance function $f(x)$ there exists a value $A = A_b = 2 - \sqrt{2}$ that plays an essential role in time-optimal structure formation. Namely, if $A \leqslant A_b$ then the time-optimal control process is typical, analogous as in classical case $\ddot{x} = u, |u| \leqslant 1$, i.e. there exists a switching curve formed by the trajectories of time-optimal solutions reaching the target state and the time-optimal process is formed by at most one switching operation. For the case $A > A_b$ we will examine two following singular phenomena.

(a) If the target state $\mathbf{z}_1 = (0, 0)$ then there exists the switching curve, dividing the state plane into two sets, however only one its branch is formed by the time-optimal solution reaching the target $\mathbf{z}_1 = (0, 0)$ and generated by the control $u = -1$. None of solution forms the second branch of switching curve. It is formed by a state-locus depending on the value of A only. In dependency of the starting state $\mathbf{z}_0$ the time-optimal control process is generated by *bang-bang* control with none, one or two switching operations. This is the *first singular phenomenon*, because any small decrease of the value A over A_b requires to change the structure which would be able to generate the time-optimal process.

(b) The paper shows, that if the target state $\mathbf{z}_1(x_1, 0), x_1 > 0$ then there exists a set of the starting states from which there start two trajectories reaching the target in the same minimum time. This is the *second phenomenon*.

Finally, some suggestions as to practical applications have been given too.

Key words: Time-optimal feedback system, singular phenomena in a time-optimal problem, global synthesis of time-optimal feedback system.

1. Introduction

Industrial devices, such as saddles of machine tools, tracer machines, industrial manipulators, several parts of industrial robots, or the position mechanisms of industrial automata, need to change their position in a minimum time, particularly when it is necessary to move the mechanism before another technological operation can proceed. Synthesis of a time-optimal control structure becomes therefore an important, economical problem.

Dynamics of the above devices, called *position mechanisms* depend essentially on motion resistance. From technical point of view we distinguish motion resistance depending on velocity of the mechanism or on its position. If the first type of that motion resistance is a case then the dynamics of the controlled object is given by [5]: $\dot{x} = y, \dot{y} = f(y) + u$, where x, y is position and velocity of the mechanism respectively, f is a function of motion resistance, u is a control function. In order to define as large as possible class of motion resistance, in particular all types of friction, we assume that function f is piecewise continuous. Discontinuity of the right-hand side of the above model makes the classical theory of differential equation, as well as the maximum principle, impossible to apply to the time-optimal problem. This problem has been solved with the use of differential inequality theory by assumption that both the control function and co-ordinates are constrained: $|y| \leqslant y_m, |\dot{y}| \leqslant \dot{y}_m$. The solution mentioned above, has been used for feedback control system creation, based on the concept of regular closed-loop system synthesis [2], [6, 7]. The closed-loop system created in such a way is operating analogously as that created for the classical type of the dynamic object: $\ddot{x} = u, |u| < 1$.

If the second type of motion resistance is a case i.e. if they are depending on the position of the mechanism only, then the dynamics of the *position mechanisms* is defined by the following differential equation: $\dot{x} = y, \dot{y} = f(x) + u$.

In this paper we will work with the following mapping of position mechanism dynamics:

$$\left.\begin{array}{ll} \dot{x} = y, & x(0) = x_0 \\ \dot{y} = f(x) + u, & y(0) = y_0 \end{array}\right\} \tag{1.1}$$

by $|u| \leqslant 1$ and motion resistance function given as follows:

$$f(x) = \begin{cases} 0, & x \leqslant 0, \\ -A, & x > 0, \quad 0 \leqslant A < 1. \end{cases} \tag{1.2}$$

The model (1.1), (1.2) describes dynamics of a very important class of industrial installations, namely manipulators with counterweight, outriggers of position mechanisms and a lot of the like devices.

The paper deals with particular cases of the time-optimal problem of the system (1.1), (1.2) that will be understood as a transfer the initial state $\mathbf{z}_0 = (x_0, y_0) \in R^2$ to the target state $\mathbf{z}_1 = (x_1, 0), x_1 \geqslant 0$ in a minimum time $t^* < \infty$. There has been shown that in the formula (1.2) defining motion resistance function there exists a value $A = A_b = 2 - \sqrt{2}$ that plays an essential role in time-optimal structure formation. Namely, if $A \leqslant A_b$ then the time-optimal control process is typical, analogous as in classical case $\ddot{x} = u, |u| \leqslant 1$. Thus, in the state plane there exists a switching curve formed by standard solutions of (1.1), (1.2) reaching the target $\mathbf{z}_1$. The control process is of *bang-bang* type and to each state belonging either to several branches of this switching curve or doing to the sets resulting from partitioning the state plane by that switching curve there are admitted the time-optimal controls $u \equiv +1$ and $u \equiv -1$. The time-optimal control process is of *bang-bang* type with at most one switching operation.

For the case $A > A_b$ we will examine two following singular phenomena.

(a) If the target state $\mathbf{z}_1 = (0,0)$ then there exists also the switching curve, dividing the state plane into two sets, however only one its branch is formed by the solution of (1.1), (1.2) reaching the target $\mathbf{z}_1 = (0,0)$ and generated by the control $u \equiv -1$. None of (1.1), (1.2) solution forms the second branch of switching curve. It is formed by a state-locus depending on the value of A only. In dependency of the starting state $\mathbf{z}_0$ the time-optimal control process is generated by *bang-bang* control with none, one or two switching operation. This is the *first singular phenomenon*, because any small decrease of the value A over A_b requires to change the structure which would be able to generate the time-optimal process.

(b) The paper shows, that if the target state $\mathbf{z}_1 = (x_1, 0), x_1 > 0$ then there exists a set of the starting states from which there start two different trajectories reaching the target in the same minimum time. This is the *second singular phenomenon*.

The desirability of implementing time-optimal feedback control in technical applications has been justified in the last paragraph of the paper. Global synthesis of the time-optimal system requires both the global uniqueness of optimal solutions and univocal defined sets of the states in which the switching operation of the control function should be executed. In investigated dynamic system (1.1), (1.2) non-unique time optimal trajectories and essential alternation of the low of time optimal control by increase of the value of parameter A over the critical value A_b unfortunately eliminate the chance of standard way of global synthesis of the time-optimal feedback system. The results of this paper indicate these resistance functions $f(x)$ for which there exists time-optimal global synthesis and for which it does not. From this work there results also that revealed *singular phenomena* may come into existence also by continuous motion resistance function $f(x)$ if only its values increase on suitable small interval of the variable x.

2. Preliminaries

NOTATIONS 2.1. (a) Any solution of (1.1), (1.2) by $u \in (-1, +1)$ starting from the initial state $\mathbf{z}_0 \in R^2$ will be denoted by $\mathbf{q}(t; \mathbf{z}_0), y(t, \mathbf{z}_0))$.

(b) The solutions of the system (1.1), (1.2) generated by the control function $u \equiv +1$ and $u \equiv -1$ starting from any point $\mathbf{z}_i$ will be denoted $\mathbf{q}_+(t; \mathbf{z}_i)$ and $\mathbf{q}_-(t; \mathbf{z}_i)$ respectively or shortly (in particular in the figures) $\mathbf{q}_+$ and $\mathbf{q}_-$.

(c) Trajectories of the solutions $\mathbf{q}_+(t; \mathbf{z}_0)$ and $\mathbf{q}_-(t; \mathbf{z}_0)$ reaching the target state $\mathbf{z}_1$ will play an essential role. They will be called *Terminal Trajectories*, will be denoted $\mathbf{T}^+$ and $\mathbf{T}^-$ respectively and will be defined by:

$$\mathbf{T}^+ = \{\mathbf{q}_+(t; \mathbf{z}_1), y \leqslant 0\} = \left\{(x, y) : x = \frac{y^2}{2(1-A)} + x_1, y \leqslant 0\right\} \tag{2.1}$$

$$\begin{aligned}\mathbf{T}^- &= \{\mathbf{q}_-(t;\mathbf{z}_1), \quad t \leqslant 0\} = \\ &\quad \left\{(x,y) : x = -\frac{y^2}{2} + (1+A)x_1, \quad y \geqslant \sqrt{2(1+A)x_1};\right. \\ x &= \left. -\frac{y^2}{2(1+A)} + x_1, \quad y \in [0, \sqrt{2(1+A)x_1}]\right\}. \qquad (2.2)\end{aligned}$$

Trajectory $\mathbf{T}^-$ intersects the positive semi-y-axis in the point noted $\mathbf{z}_y = (0, y_y)$, where $y_y = \sqrt{2(1+A)x_1}$. If the target $\mathbf{z}_1 = (0,0)$ then, after setting $0 \to x_1$, the formulas (2.1), (2.2) take the forms:

$$\begin{aligned}\mathbf{T}^+ &= \left\{(x,y) : x = \frac{y^2}{2(1-A)}, y \leqslant 0\right\}, \\ \mathbf{T}^- &= \left\{(x,y) : x = -\frac{y^2}{2}, y \geqslant 0\right\}.\end{aligned} \qquad (2.3)$$

(d) The negative and positive semi-y-axes will be noted respectively

$$\mathbf{B}^- = \{(x,y) : x = 0, y \geqslant 0\}; \qquad \mathbf{B}^+ = \{(x,y) : x = 0, y \leqslant 0\}. \qquad (2.4)$$

The y-axis $\mathbf{B} = \mathbf{B}^- \cup \mathbf{B}^+$ forms the *bound of motion resistance zone* and divides the state-plane into two following half-planes:

$$\mathbf{S}^- = \{(x,y) : x < 0, y \in R^1\}; \qquad \mathbf{S}^+ = \{(x,y) : x > 0, y \in R^1\}. \qquad (2.5)$$

(e) A time taken for transfer the state along a trajectory of any solution starting from a point $\mathbf{z}'$ and running over the points $\mathbf{z}'', \mathbf{z}''', \ldots$ to finite one $\mathbf{z}^i$ will be denoted $T(\mathbf{z}', \mathbf{z}'', \mathbf{z}''', \ldots, \mathbf{z}^i)$.

(f) Co-ordinates of any state $\mathbf{z}_i$ will be denoted x_i and y_i, i.e. $\mathbf{z}_i = (x_i, y_i)$. ■

REMARK 2.2. *Properties of the solutions* $\mathbf{q}_-$ *and* $\mathbf{q}_+$.

(a) The co-ordinates of the solution $\mathbf{q}_-(t;\mathbf{z}_0)$ has got the following properties. Let $y_0 > 0$. Then, there exists a time $t_1 > 0$ such that $y_-(t, \mathbf{z}_0)$ is decreasing function on $[0, \infty), y_1(t_1, \mathbf{z}_0) = 0$, but $x_-(t, \mathbf{z}_0)$ is increasing function on $[0, t_1]$ and is decreasing one on $[t_1, \infty)$.

(b) The co-ordinates of the solution $\mathbf{q}_+(t;\mathbf{z}_0)$ has got the following properties. Let $y_0 < 0$. Then, there exists a time $t_1 > 0$ such that $y_+(t, \mathbf{z}_0)$ is increasing function on $[0, t_1], y_+(t_1, \mathbf{z}_0) = 0$, but $x_+(t, x_0)$ is decreasing function on $[0, t_1]$ and increasing one on $[t_1, \infty)$. ■

LEMMA 2.3. *Given controlled object (1.1), (1.2). The time-optimal control* u^* *bringing the controlled object from any* $\mathbf{z}_0 \in R^2$ *to the target state* $\mathbf{z}_1 = (x_1, 0), 0 \leqslant x_1$ *is of* bang-bang type, *i.e. the control function* $u \equiv +1$ *and* $u \equiv -1$.

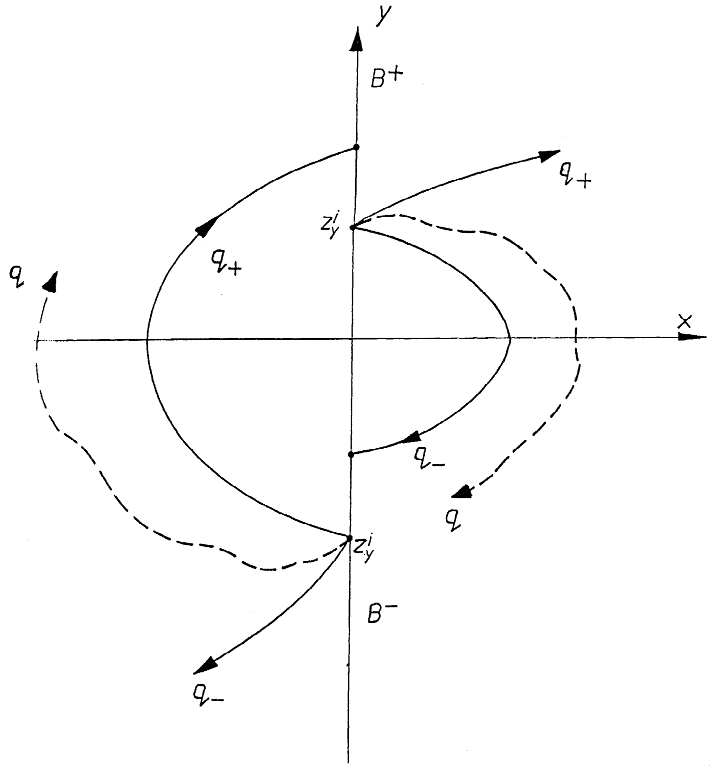

Figure 1. Trajectories starting from $\mathbf{B}^+$ and $\mathbf{B}^-$ sets

Proof. Assume the time-optimal solution of (1.1), (1.2) does exist. Let the trajectory of a time-optimal solution $\mathbf{q}^*(t;\mathbf{z}_0)$ starting from any $\mathbf{z}_0 \in R^2$ and reaching the target $\mathbf{z}_1$ runs over the state plane intersecting y-axis finite number of times in the points t_i, $i = 1, 2, \ldots, f$. Denote $\mathbf{z}_y^i = \mathbf{q}^*(t_i; \cdot)$. Thus, the $\mathbf{z}_y^1$ and $\mathbf{z}_y^f$ denote the first and the last state in which trajectory of time-optimal solution $\mathbf{q}^*$ intersects y-axis. From Remark 2.2 it follows, if $\mathbf{z}_y^i \in \mathbf{B}^+$[or $\mathbf{z}_y^i \in \mathbf{B}^-$] then $\mathbf{z}_y^{i+1} \in \mathbf{B}^-$[or $\mathbf{z}_y^{i+1} \in \mathbf{B}^+$]. Analogously, if $\mathbf{z}_y^i \in \mathbf{B}^+$[or $\mathbf{z}_y^i \in \mathbf{B}^-$] then $\mathbf{z}_y^{i-1} \in \mathbf{B}^-$[or $\mathbf{z}_y^{i-1} \in \mathbf{B}^+$] (see Figure 1). Thus, the time-optimal trajectory starting from $\mathbf{z}_y^f$ with the target $\mathbf{z}_1$ lies totally in half-plane $\mathbf{S}^+$. Using *Maximum Principle* in standard way we state that the last part of time-optimal trajectory, i.e. trajectory connecting $\mathbf{z}_y^f$ with the target $\mathbf{z}_1$ is of *bang-bang* type. In the same way of argument we prove that the first part of time-optimal trajectory, i.e. trajectory connecting $\mathbf{z}_0$ with $\mathbf{z}_y^1$ is of *bang-bang* type, too. Obviously, if $\mathbf{z}_y^1 \in \mathbf{B}^+$ then $\mathbf{z}_y^1 \equiv \mathbf{z}_y^f$, therefore we should investigate the case $\mathbf{z}_y^1 \in \mathbf{B}^-$. Using the *Maximum Principle* in standard way as that in the case $\ddot{x} = u, |u| \leqslant 1$ for starting point $\mathbf{z}_0 \in \mathbf{B}^-$ and the target $\mathbf{z}_1 \in \mathbf{B}^+$ we state that the time-optimal trajectory generated by the system (1.1), (1.2) starting from $\mathbf{z}_y^1 \in \mathbf{B}^-$ and reaching $\mathbf{z}_y^f \in \mathbf{B}^+$ is of the *bang-bang* type and lies totally in the set $\mathbf{S}^-$. This completes the proof. ■

3. Dependence between Number of Time-Optimal Switching Operations and Parameter A Value

LEMMA 3.1. *Given controlled object (1.1), (1.2) and terminal trajectory* $\mathbf{T}^-$ *(2.2). Then, from ech* $\mathbf{z}_0 \in \mathbf{T}^-$ *there starts the unique solution* $\mathbf{q}_-(t; \mathbf{z}_0)$ *that lies totally in terminal trajectory* $\mathbf{T}^-$ *and reaches the target* $\mathbf{z}_1$ *in a minimum time* $t^* < \infty$.

Proof. Going by Lemma 2.1 we will examine the solutions generated by the *bang-bang* control function u only.

A time taken for the transfer the controlled object from $\mathbf{z}_0 \in \mathbf{T}^- \cap \mathbf{S}^-$ to the target $\mathbf{z}_1$ along the terminal trajectory $\mathbf{T}^-$ (i.e. by the solution $\mathbf{q}_-(t; \mathbf{z}_0)$)

$$T(\mathbf{z}_0, \mathbf{z}_y, \mathbf{z}_1) = y_0 - \frac{A\sqrt{2(1+A)x_1}}{1+A}. \tag{3.1}$$

Assume there exists another trajectory that reaches the target $\mathbf{z}_1$ in a time t' less than minimum one, i.e. $t' < t^* < \infty$. This trajectory being of *bang-bang* type should intersect semi-y-axis $\mathbf{B}^+$ in the point $\mathbf{z}_w = (0, y_w)$ such that $y_y < y_w$. The point $\mathbf{z}_w \in \mathbf{B}^+$ may be reached along the trajectory of the $\mathbf{q}_+(t; \mathbf{z}_0)$ solution to the point $\mathbf{z}_s = (0, y_s) \in \mathbf{S}^-$, where there is executed switching operation (see Figure 2). Simple computing shows that

$$y_s \in (\sqrt{2(1+A)x_1}, \sqrt{2y_0^2 - 2(1+A)x_1}) \tag{3.2}$$

where

$$y_0 \geqslant y_y = \sqrt{2(1+A)x_1}. \tag{3.3}$$

So, from the point $\mathbf{z}_s$ there starts the trajectory of $\mathbf{q}_-(t; \mathbf{z}_s)$ solution which brings the system to the point $\mathbf{z}_w = (0, y_w) \in \mathbf{B}^+$. Obviously, $y_w < y'$. The trajectory of the $\mathbf{q}_-(t; \mathbf{z}_s)$ solution starting from $\mathbf{z}_s \in \mathbf{S}^-$ intersects semi-y-axis $\mathbf{B}^+$ in the point $\mathbf{z}_w \in \mathbf{B}^+$, penetrates into $\mathbf{S}^-$ set, intersects x-axis in the point $\mathbf{z}'^w = (x'_w, 0)$ and next reaches a state $\mathbf{z}''_w = (x_1, y''_w)$ (see Figure 2). From uniqueness of the solutions $\mathbf{q}_+$ and $\mathbf{q}_-$ it follows that $x_1 < x'_w$. A time taken for a transfer the system from $\mathbf{z}_0 \in \mathbf{T}^- \cap \mathbf{S}^-$ to $\mathbf{z}''_w$ over switching point $\mathbf{z}_s \in \mathbf{S}^-$, and next over $\mathbf{z}_w$ and $\mathbf{z}'_w$ is given by expression:

$$T(\mathbf{z}_0, \mathbf{z}_s, \mathbf{z}_w, \mathbf{z}'_w, \mathbf{z}''_w) = \sqrt{2y_w^2 + 4y_0^2 - 4(1+A)x_1} - y_0 - y_w + \frac{y_w + \sqrt{y_w^2 - 2(1+A)x_1}}{1+A}. \tag{3.4}$$

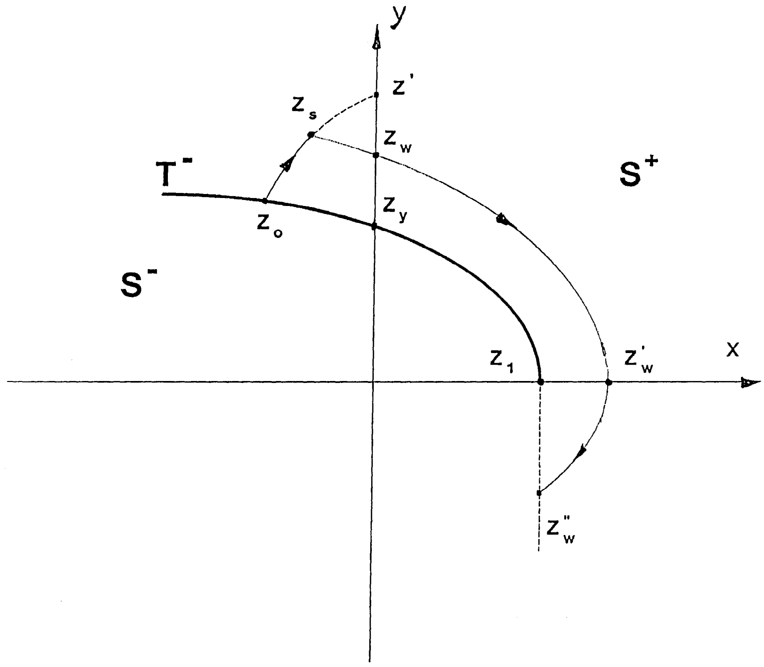

Figure 2. Trajectories starting from curve $\mathbf{T}^-$

Comparing (3.1) with (3.4) we get

$$T(\mathbf{z}_0, \mathbf{z}_s, \mathbf{z}_w, \mathbf{z}'_w, \mathbf{z}''_w) - T(\mathbf{z}_0, \mathbf{z}_w, \mathbf{z}_1) =$$

$$\sqrt{x_1^2 + 2x_1\sqrt{2(1+A)x_1}} - Ax_1 \geqslant x_1 - Ax_1 \geqslant 0. \tag{3.5}$$

This means that dynamic object (1.1), (1.2) starting from $\mathbf{z}_0 \in \mathbf{T}^- \cap \mathbf{S}^-$ is brought to the target $\mathbf{z}_1$ in minimum time along the terminal trajectory $\mathbf{T}^-$. If the switching operation is executed in the point $\mathbf{z}_s \in \mathbf{S}^+$ then the time taken for the transfer of the system from $\mathbf{z}_0 \in \mathbf{T}^- \cap \mathbf{S}^-$ to the adequate state point $\mathbf{z}''_w$ is longer than the above calculated by (3.4). The way of argument is trivial. This completes the proof of Lemma. ■

Now, we are going to investigate time-optimal solution of the object (1.1), (1.2) for selected both starting point $\mathbf{z}_0$ and the target $\mathbf{z}_1$.

3.1. TIME-OPTIMAL PROBLEM FOR THE TARGET STATE $\mathbf{z}_1 = (0, 0) = \mathbf{0}$

If the target state $\mathbf{z}_1 = \mathbf{0}$ is a case then the terminal trajectories are defined by (2.3). Minimum time taken for the transfer the state $\mathbf{z}_0 \in \mathbf{T}^-$ to the target $\mathbf{z}_1 = \mathbf{0}$ (obviously along the terminal trajectory $\mathbf{T}^-$) results from (3.1) after setting $x_1 = 0$ and is expressed by: $T(\mathbf{z}_0, \mathbf{z}_1) = y_0$.

At first we will examine the time-optimal trajectories starting from $\mathbf{z}_0 \in \mathbf{T}^+$. There will be distinguished two following cases of the motion resistance function:

$$\text{(i)} \quad A \in [0, 2-\sqrt{2}]; \qquad \text{(ii)} \quad A \in (2-\sqrt{2}, 1). \tag{3.6}$$

Now, we are going to show that if motion resistance function satisfies (3.6, i) then there exists the time-optimal switching curve $\mathbf{T} = \mathbf{T}^+ \cup \mathbf{T}^-$ where $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined by (2.3). However, if motion resistance function satisfies (3.6, ii) then there exists the time-optimal switching curve $\mathbf{T} = \mathbf{T}_m \cup \mathbf{T}^-$ where $\mathbf{T}^-$ is given by (2.3). The branch $\mathbf{T}_m$ is a special state locus which cannot be created by whatever solution of the system (1.1), (1.2). It will be defined in what follows. The switching curves shown above play the same role as that in classical time-optimal closed-loop system controlling the dynamic object described by: $\ddot{x} = u, |u| \leqslant 1$.

LEMMA 3.2. *Given the controlled object (1.1), (1.2). Let starting state $\mathbf{z}_0 \in \mathbf{T}^+$ and the target state $\mathbf{z}_1 = (0,0) = \mathbf{0}$.*
Thesis (a) If $A \in (2-\sqrt{2}, 1)$ then the transfer of the object from $\mathbf{z}_0 \in \mathbf{T}^+$ to the target $\mathbf{z}_1 = \mathbf{0}$ in minimum time $t^* < \infty$ is performed along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution to a state $\mathbf{z}_m \in \mathbf{S}^+$, afterwards along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution to a state $\mathbf{z}_m \in \mathbf{S}^+$, afterwards along the trajectory of the $\mathbf{q}_+(t;\mathbf{z}_m)$ solution to $\mathbf{z}_n \in \mathbf{T}^-$ and finally from $\mathbf{z}_n$ along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$ (see Figure 3).
Thesis (b) If $A \in [0, 2-\sqrt{2}]$ then the transfer of the object from $\mathbf{z}_0 \in \mathbf{T}^+$ to the target $\mathbf{z}_1$ in minimum time $t^* < \infty$ is performed along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution, i.e. along the curve $\mathbf{T}^+$.

Proof. A way of argument will be grounded on Lemma 3.1 and Remark 2.2.

Starting from $\mathbf{z}_0 \in \mathbf{T}^+$ the object may be brought to the target $\mathbf{z}_1$ either along the curve $\mathbf{T}^+$ or along any other trajectory of $\mathbf{q}_-$ solution which on leaving $\mathbf{T}^+$ runs over $\mathbf{S}^+$ set and tends to intersect y-axis in the point $\mathbf{z}_p \in \mathbf{B}^-$ (see Figure 3).

As time-optimal trajectories starting from $\mathbf{z}_0 \in \mathbf{T}^+$ may be taken into account either trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution that lies totally on the curve $\mathbf{T}^+$ or the trajectory created consecutively by the following solutions: $\mathbf{q}_-(t;\mathbf{z}_0)$ starting from $\mathbf{z}_0 \in \mathbf{T}^+$ and reaching $\mathbf{z}_m \in \mathbf{S}^+ \cup \mathbf{B}^-$, $\mathbf{q}_+(t;\mathbf{z}_m)$ starting from $\mathbf{z}_m$ and reaching $\mathbf{z}_n \in \mathbf{T}^-$ and finally $\mathbf{q}_-(t;\mathbf{z}_n)$ reaching the target $\mathbf{z}_1$ along the curve $\mathbf{T}^-$ (see Figure 3).

Time taken for the transfer $\mathbf{z}_0 \in \mathbf{T}^+$ to $\mathbf{z}_1$ along $\mathbf{T}^+$ is expressed by:

$$T(\mathbf{z}_0, \mathbf{z}_1) = -\frac{y_0}{1-A}. \tag{3.7}$$

Let us consider the trajectory of the solution $\mathbf{q}_-(t;\mathbf{z}_0), \mathbf{z}_0 \in \mathbf{T}^+$. This trajectory intersects y-axis in a point $\mathbf{z}_y = (0, y_y) \in \mathbf{B}^-$ where $y_y = y_0\sqrt{2/(1-A)}$.

Denote by $\mathbf{z}_m = (x_m, y_m)$ the states belonging to this trajectory between $\mathbf{z}_0$ and $\mathbf{z}_r = (0, y_r)$. This means that the states $\mathbf{z}_m$ laid in the region determined by the curve $\mathbf{T}^+$ and semi-axes $\mathbf{B}^-$ (see Figure 3). The co-ordinates of the states $\mathbf{z}_m$ satisfy the following inequalities: $0 \leqslant x_m \leqslant x_0$ and $y_y \leqslant y_m \leqslant 0$ (see Figure 3).

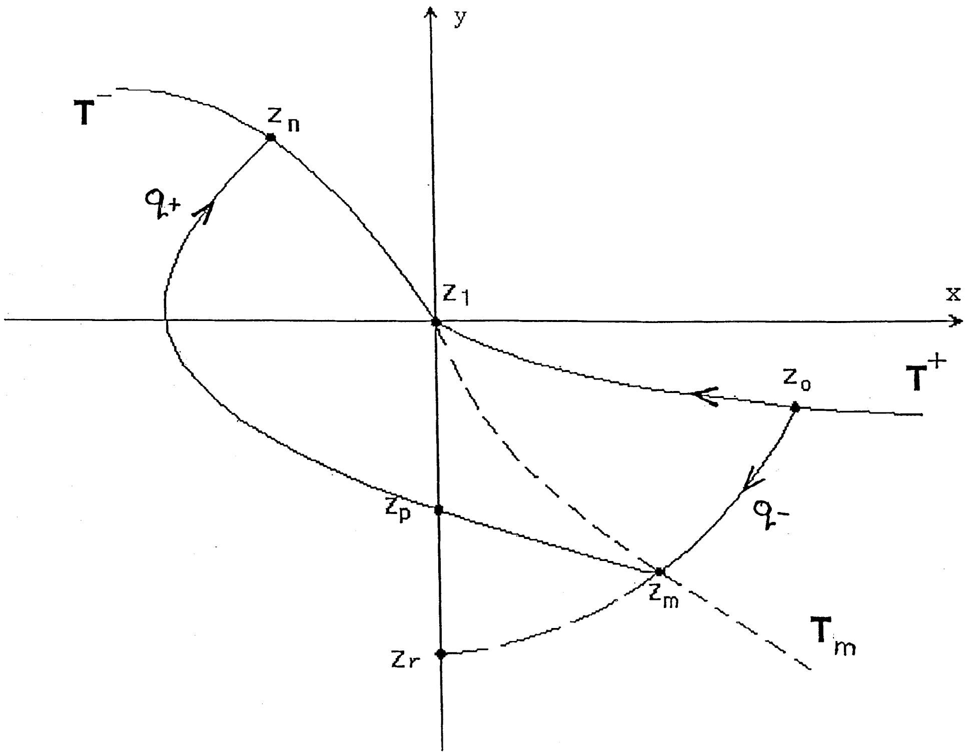

Figure 3. Switching curve $\mathbf{T}_m$

Trajectory of $\mathbf{q}_+(t;\mathbf{z}_m)$ solution, i.e. trajectory starting from $\mathbf{z}_m$ intersects semi-axis $\mathbf{B}^-$ in the point $\mathbf{z}_p = (0, y_p)$ where $y_p \in [y_r, 0]$. Now, we are going to express the co-ordinates of the states $\mathbf{z}_m = (x_m, y_m)$ as the functions of y_p. We get:

$$x_m = \frac{2y_0 - (1-A)y_p^2}{4(1-A)}, \qquad y_m = -\sqrt{\frac{2y_0^2 + (1+A)y_p^2}{2}}. \tag{3.8}$$

Time taken for the transfer the object from the state $\mathbf{z}_0 \in \mathbf{T}^+$ to any $\mathbf{z}_m$ along trajectory of $\mathbf{q}_-$ solution, from $\mathbf{z}_m$ over $\mathbf{z}_p$ to $\mathbf{z}_n \in \mathbf{T}^-$ along the trajectory of $\mathbf{q}_+$ solution and finally from $\mathbf{z}_n$ along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$ is given by the following formula:

$$T(\mathbf{z}_0, \mathbf{z}_m, \mathbf{z}_p, \mathbf{z}_n, \mathbf{z}_1) =$$

$$\frac{\sqrt{2(1+A)y_p^2 + 4y_0^2} - y_p(1+A)(\sqrt{2} - A\sqrt{2})}{1-A} + \frac{y_0}{1+A}. \tag{3.9}$$

Values of time defined by (3.6) and (3.9) depend on A, y_0 and y_p variables only. Thus, a difference that may exist between them may be expressed as a function of the same variable. That difference will be noted $\Delta T(A, y_p, y_0)$. It is given by:

$$\Delta T(A, y_p, y_0) = T(\mathbf{z}_0, \mathbf{z}_m, \mathbf{z}_p, \mathbf{z}_n \mathbf{z}_1) - T(\mathbf{z}_0, \mathbf{z}_1) =$$
$$\frac{\sqrt{2(1+A)y_p^2 + 4y_0^2} - y_p(1+A)(\sqrt{2} - A - A\sqrt{2}) + 2y_0}{1 - A^2}. \tag{3.10}$$

Now, we are going to show that there exists a state $\mathbf{z}_p = (0, y_p) = \bar{\mathbf{z}}_p = (0, \bar{y}_p)$ such that

$$\Delta T(A, \bar{y}_p, y_0) = \min_{y_p \in [y_r, 0]} \Delta T(A, \bar{y}_p, y_0). \tag{3.11}$$

Let us denote derivative of $\Delta T(A, y_p, y_0)$ towards y_p by:

$$\Delta T'(A, y_p, y_0) = \frac{\partial \Delta T(A, y_p, y_0)}{\partial y_p}. \tag{3.12}$$

From (3.10), (3.11) we get

$$\Delta T'(A, y_p, y_0) = \frac{1}{1-A}\left[\sqrt{2} - A - A\sqrt{2} - \frac{2y_p}{\sqrt{2(1+A)y_p^2 + 4y_0^2}}\right]. \tag{3.13}$$

After solving equation $\Delta T'(A, y_p, y_0) = 0$ we get

$$\frac{2y_p}{\sqrt{2(1+A)y_p^2 + 4y_0^2}} = \sqrt{2} - A - A\sqrt{2}. \tag{3.14}$$

From the above it follows that the values of y_p which may minimise the function $\Delta T(A, \bar{y}_p, y_0)$ are defined by:

$$y_p = \frac{y_0(2A + A\sqrt{2} - 2)}{\sqrt{A(1-A)(3A + 2A\sqrt{2} + 2 + 2\sqrt{2})}}. \tag{3.15}$$

Simple estimation shows that

$$\frac{2A + A\sqrt{2} - 2}{\sqrt{A(1-A)(3A + 2A\sqrt{2} + 2 + 2\sqrt{2}}} > \sqrt{\frac{2}{1-A}}$$

which means that $y_r < y_p$.

Expression (3.11), by statement that co-ordinate $y_p \in [y_r, 0]$ and $y_r < 0$ imply the following conclusions:

(I) If only $\sqrt{2} - A - A\sqrt{2} < 0$, i.e. if $A \in (2 - \sqrt{2}, 1)$ then the function (3.11) takes its minimum by $\bar{y}_p \in [y_r, 0)$. In other words

$$\bar{\mathbf{z}}_p = (0, \bar{y}_p), \bar{y}_p \in [y_r, 0) \quad \text{if only } A \in (2 - \sqrt{2}, 1)$$

(ii) If only $\sqrt{2} - A - A\sqrt{2} \geqslant 0$, i.e. if $A \in [0, 2 - \sqrt{2}]$ then the function (3.11) takes its minimum by $y_p = \bar{y}_p = 0$. In other words

$$\bar{\mathbf{z}}_p \equiv \mathbf{z}_1 = (0, 0) \quad \text{if only } A \in [0, 2 - \sqrt{2}]$$

Conclusions (i) and (ii) complete the proof of Theses (a) and (b) respectively. ∎

Let us perceive that if $A \in (2 - \sqrt{2}, 1)$ is a case then the locus of the states $\mathbf{z}_m$ forms a switching curve noted (in accordance with Figure 3) $\mathbf{T}_m$. Using expressions (3.8), (3.15) we define the switching curve $\mathbf{T}_m$ by the following formula:

$$\mathbf{T}_m = \left\{ (x, y) : x = \frac{2 - (1 - A)B^2}{2(1 - A)[2 + (1 + A)B^2]}\, y^2, y \in [y_r, 0] \right\} \tag{3.16}$$

where

$$B = \frac{2A + A\sqrt{2} - 2}{\sqrt{A(1 - A)(3A + 2A\sqrt{2} + 2 + 2\sqrt{2})}} \tag{3.17}$$

and as previously $y_r = y_0\sqrt{2/1 - A}, y_0 \leqslant 0$.

It should be emphasized that the switching curve $\mathbf{T}_m$ is none of the trajectories which may be formed by any one solution of (1.1), (1.2) (see Figure 3).

REMARK 3.3. From Lemma 3.2 it follows that if motion resistance function satisfies inequality (3.6, i), i.e. $A \in [0, 2 - \sqrt{2}]$ then the time-optimal transfer of each state $\mathbf{z}_0 \in \mathbf{T}^+$ holds along the trajectory of $\mathbf{q}_+(t; \mathbf{z}_0)$ solution, i.e. along the curve $\mathbf{T}^+$ without any switching of the control function u. However, if $A \in (2 - \sqrt{2}, 1)$ then the time-optimal transfer of the object from $\mathbf{z}_0 \in \mathbf{T}^+$ there starts the trajectory of the $\mathbf{q}_-(t; \mathbf{z}_0)$ solution which reaches the state $\mathbf{z}_m \in \mathbf{T}_m \subset \mathbf{S}^-$ where the switching operation is being executed. From $\mathbf{z}_m$ there starts the trajectory of the $\mathbf{q}_+(t; \mathbf{z}_m)$ solution the trajectory of which intersects semi-y-axis $\mathbf{B}^-$, penetrates into $\mathbf{S}^-$ set and tends to reach the curve $\mathbf{T}^-$ in the point $\mathbf{z}_n \in \mathbf{T}^- \cap \mathbf{S}^-$ where there should be executed the second switching operation. From $\mathbf{z}_n$ there starts the trajectory of $\mathbf{q}_-(t; \mathbf{z}_0)$ solution which brings the object along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1 = \mathbf{0}$. ∎

Let us define in the state plane some special sets for the cases of motion resistance function quoted above, that will be of use in the next part of the text.

(1) If $A \in [0, 2 - \sqrt{2}]$ then we will note (see Figure 4):

$$\mathbf{T}_{\mathrm{I}} = \mathbf{T}^- \cup \mathbf{T}^+ \tag{3.18}$$

$$\mathbf{R}_{\mathrm{I}}^- = \{(x, y) : (x', y) \in \mathbf{T}_{\mathrm{I}} \Rightarrow x > x'\} \tag{3.19}$$

$$\mathbf{R}_{\mathrm{I}}^- = \{(x, y) : (x', y) \in \mathbf{T}_{\mathrm{I}} \Rightarrow x > x'\} \tag{3.20}$$

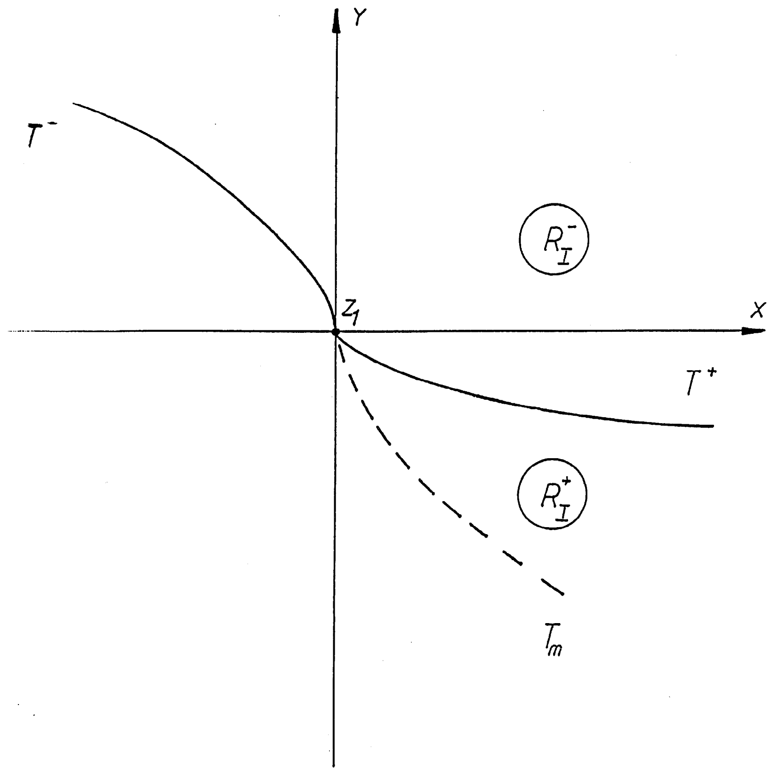

Figure 4. State plane partitioning

(2) If $A \in (2 - \sqrt{2}, 1)$ then we will note (see Figure 5):

$$\mathbf{T}_{\mathrm{II}} + \mathbf{T}^- \cup \mathbf{T}_m \tag{3.21}$$

$$\mathbf{R}_{\mathrm{II}}^+ = \{(x, y) : (x', y) \in \mathbf{T}_{\mathrm{II}} \Rightarrow x < x'\} \tag{3.22}$$

$$\mathbf{R}_{\mathrm{II}}^- = \{(x, y) : (x', y) \in \mathbf{T}_{\mathrm{II}} \Rightarrow x > x'\} \tag{3.23}$$

THEOREM 3.4. *Given controlled object (1.1), (1.2) and target state* $\mathbf{z}_1 = (0, 0) = \mathbf{0}$. *Thesis (a). If the motion resistance function (1.2) satisfies inequality (3.6, i), i.e.* $A \in [0, 2 - \sqrt{2}]$ *then the time-optimal control function*

$$u^*(x, y) = \left| \begin{array}{l} +1, \ (x, y) \in \mathbf{T}^+ \cup \mathbf{R}_{\mathrm{I}}^+ \\ -1, \ (x, y) \in \mathbf{T}^- \cup \mathbf{R}_{\mathrm{I}}^- \end{array} \right. \tag{3.24}$$

where $\mathbf{T}^+, \mathbf{T}^-, \mathbf{R}_{\mathrm{I}}^+, \mathbf{R}_{\mathrm{I}}^-$ *are defined by (2.3), (3.19) and (3.20) respectively.*
Thesis (b). If the motion resistance function (1.2) satisfies inequality (3.6, ii), i.e. $A \in (2 - \sqrt{2}, 1)$ *then the time-optimal control function*

$$u^*(x, y) = \left| \begin{array}{l} +1, \ (x, y) \in \mathbf{T}_m \cup \mathbf{R}_{\mathrm{II}}^+ \\ -1, \ (x, y) \in \mathbf{T}^- \cup \mathbf{R}_{\mathrm{II}}^- \end{array} \right. \tag{3.25}$$

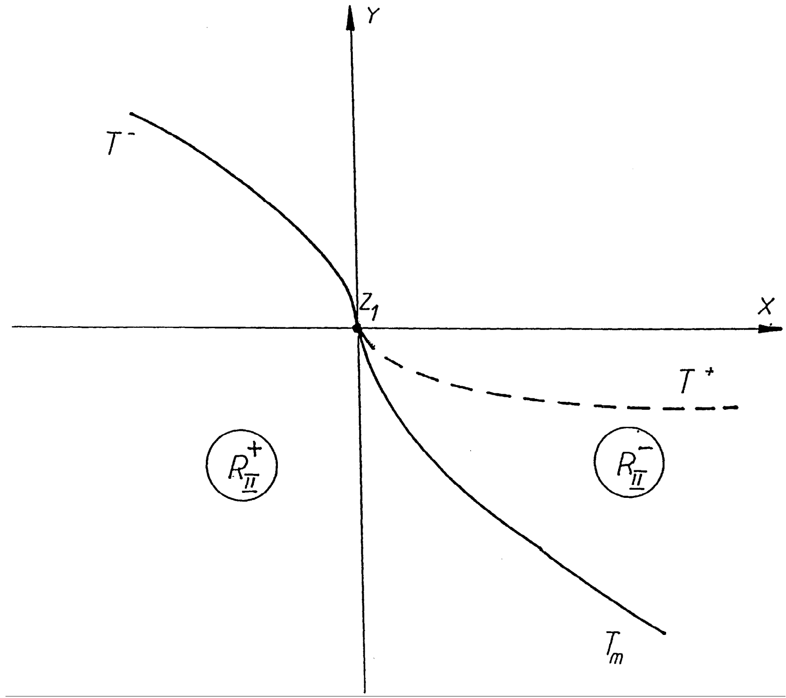

Figure 5. State plane partitioning

where $\mathbf{T}^-, \mathbf{T}_m, \mathbf{R}_{\mathrm{II}}^+$ *and* $\mathbf{R}_{\mathrm{II}}^-$ *are defined by (2.3), (2.16), (3.22) and (3.23) respectively.*

Proof of Thesis (a):

(i) If $\mathbf{z}_0 \in \mathbf{T}_{\mathrm{I}} = \mathbf{T}^- \cup \mathbf{T}^+$ then the proof results from Lemma 2.3 and *Thesis (b)* in Lemma 3.2.

(ii) From each initial state $\mathbf{z}_0 \in \mathbf{R}_{\mathrm{I}}^-$ there starts the unique trajectory of $\mathbf{q}_-(t; \mathbf{z}_0)$ solution reaching positive semi-x-axis in the point $\mathbf{z}_1' = (x_1', 0)$. Uniqueness of $\mathbf{q}_-$ solutions implies that this trajectory lies totally in the set $\mathbf{R}_{\mathrm{I}}^-$ and $x_1 < x_1'$. Therefore, the minimum time taken for the transfer of the object from $\mathbf{z}_0 \in \mathbf{R}_{\mathrm{I}}^-$ to the state $\mathbf{z}_1'$ holds along the trajectory of $\mathbf{q}_-(t; \mathbf{z}_0)$ solution. Any other trajectory starting from the same $\mathbf{z}_0 \in \mathbf{R}_{\mathrm{I}}^-$ and composed by the sequence of *bang-bang* solutions $\mathbf{q}_+$ and $\mathbf{q}_-$ reaches x-axis in the point $\mathbf{z}_1'' = (x_1'', 0)$ where $x_1' < x_1''$. Using the same concept of argument as that done in the proof of Lemma 3.1 we state that the time taken for the transfer the object from $\mathbf{z}_0$ to $\mathbf{z}_1'$ and from $\mathbf{z}_0$ to $\mathbf{z}_1''$ satisfy the following inequalities:

$$T(\mathbf{z}_0, \mathbf{z}_1') < T(\mathbf{z}_0, \ldots, \mathbf{z}_1''), \tag{3.26}$$

Elementary analysis shows that minimum time transfer of the object from $\mathbf{z}_1'$ to the curve $\mathbf{T}^+$ should be executed along the trajectory of $\mathbf{q}_-(t; \mathbf{z}_1')$ solution.

Analogously minimum time transfer of the object from $\mathbf{z}_1''$ to the curve $\mathbf{T}^+$ should be executed along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_1'')$. Calculating the intervals of the time taken for the transfer the object from $\mathbf{z}_1'$ to $\mathbf{z}_+' \in \mathbf{T}^+$ along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_1')$ and from $\mathbf{z}_1''$ to $\mathbf{z}_+'' \in \mathbf{T}^+$ along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_1'')$ we get:

$$T(\mathbf{z}_1', \mathbf{z}_+') < T(\mathbf{z}_1'', \mathbf{z}_+''). \tag{3.27}$$

Inequalities (3.26) and (3.27) imply that the system starting from $\mathbf{z}_0 \in \mathbf{R}_\mathrm{I}^-$ reaches the curve $\mathbf{T}^+$ in minimum time along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_0)$ solution.

(iii) For the case $\mathbf{z}_0 \in \mathbf{R}_\mathrm{I}^+$ the way of proving is the same as that used in the previous case (ii).

Proof of Thesis (b): We prove this thesis using the same pattern of argument as that employed in the proof of Thesis (a). ■

REMARK 3.5. Theses in Theorem 3.4 define the control function $u(x, y)$ operating in a time-optimal closed-loop system synthesised in standard way. If resistance function satisfies (4.6, i) then this system executes at most one switching operation, however if (3.6, ii) holds then this system should be able to execute at most two switching operations. Thus, any small increase of the parameter A over the value $2 - \sqrt{2}$ requires to change the nature of the closed-loop system generating time-optimal processes. This properties of the time-optimal process will be called *singular phenomenon*. ■

3.2. TIME-OPTIMAL PROBLEM FOR THE TARGET STATE $\mathbf{z}_1 = (x_1, 0), x_1 > 0$

For this case of the target state $\mathbf{z}_\mathrm{I}$ the terminal trajectories are defined by (2.1), (2.2). In this chapter we will examine the time-optimal trajectories starting from $\mathbf{z}_0 = (0,0) = \mathbf{0}$. As previously, there will be distinguished two cases of motion resistance function defined by (4.6).

LEMMA 3.6. *Given the controlled object (1.1), (1.2). Let starting state $\mathbf{z}_0 = \mathbf{0}$ and the target state $\mathbf{z}_1 = (x_1, 0), x_1 > 0$.*

Thesis (a) If $A \in (2 - \sqrt{2}, 1)$ then the transfer the object from starting state $\mathbf{z}_0 = \mathbf{0}$ to the target $\mathbf{z}_1$ in minimum time $t^* < \infty$ holds along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution to a certain state $\mathbf{z}_s \in \mathbf{S}^-$, next along the trajectory of the $\mathbf{q}_+(t;\mathbf{z}_s)$ solution over the point $\mathbf{z}_w = (0, y_w) \in \mathbf{B}^+$ to $\mathbf{z}_n = (x_n, y_n) \in \mathbf{T}^-$ and finally from $\mathbf{z}_n$ along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$ (see Figure 6).

Thesis (b) If $A \leqslant 2 - \sqrt{2}$ then the transfer the object from $\mathbf{z}_0$ to the target $\mathbf{z}_1$ in minimum time $t^* < \infty$ is performed along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution to a point $\mathbf{z}_n \in \mathbf{T}^-$ and finally from $\mathbf{z}_n$ along $\mathbf{T}^-$ curve to the target $\mathbf{z}_1$ (see Figure 7).

Proof. The proof bases on Lemma 2.3 and the properties of $\mathbf{q}_-$ and $\mathbf{q}_+$ solutions shown in Remark 2.2.

Starting from $\mathbf{z}_0 = \mathbf{0}$ the object may be brought to the target $\mathbf{z}_1$ either:

(i) along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_0)$ solution running over $\mathbf{S}^+$ set till to intersection with $\mathbf{T}^-$ in a point $\mathbf{z}_n \in \mathbf{T}^-$ and next the object is directly transferred to the

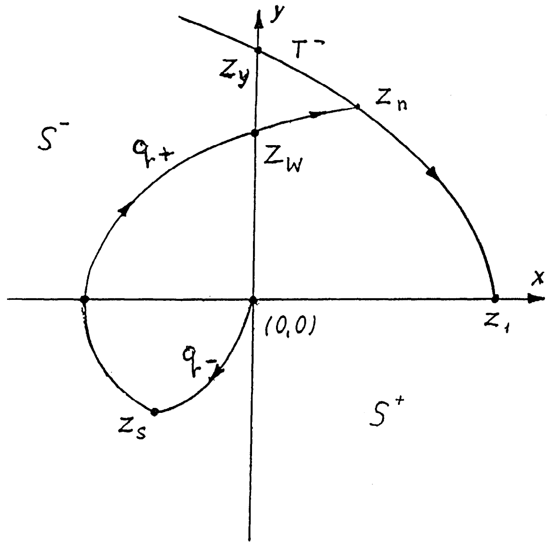

Figure 6. Trajectories starting from origin $(0, 0)$

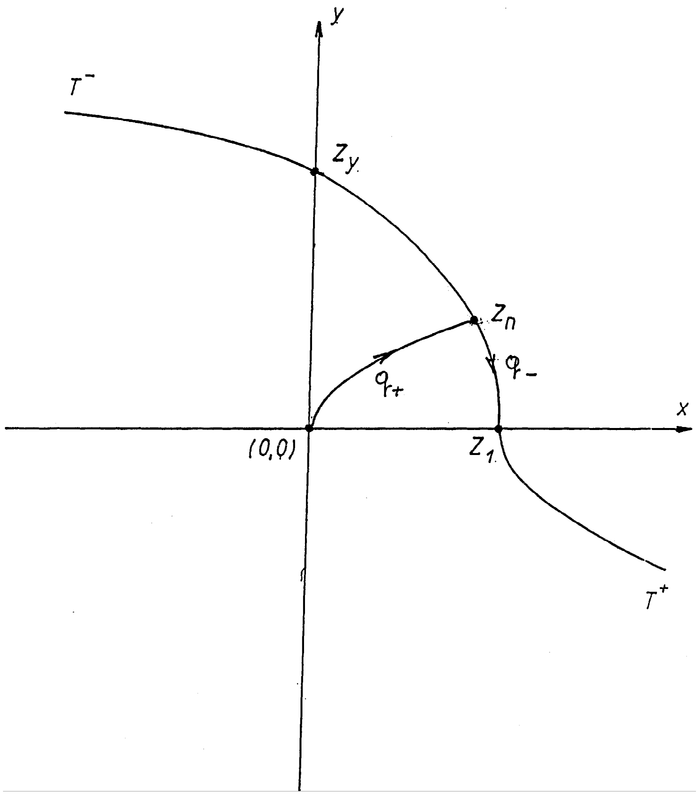

Figure 7. Trajectory intersecting switching curve $\mathbf{T}^-$

target $\mathbf{z}_1$ along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_n)$ solution that lies totally in the curve $\mathbf{T}^-$ (see Figure 7) or

(ii) along the trajectory of $\mathbf{q}_((t;\mathbf{z}_0)$ to a certain point $\mathbf{z}_s \in \mathbf{S}^-$, next from $\mathbf{z}_s$ along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_s)$ solution which running over $\mathbf{S}^-$ set intersects y-axis in the point $\mathbf{z}_w \in \mathbf{B}^+$ and next penetrating into $\mathbf{S}^+$ set reaches $\mathbf{T}^-$ curve in the point $\mathbf{z}_n \in \mathbf{T}^-$, where $y_w < y_n$. From $\mathbf{z}_n$ the object is directly transferred to the target along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_n)$ solution that lies totally on the curve $\mathbf{T}^-$ (see Figure 6).

We are now going to define some special elements in the state plane which will be of use on the way of proving.

The curve $\mathbf{T}^-$ intersects the y-axis in the point $\mathbf{z}_y = (0, y_y) \in \mathbf{B}^+$, where

$$y_y = \sqrt{2(1+A)x_1}. \tag{3.28}$$

A time taken for the transfer the object from any state $\mathbf{z} = (x, y) \in \mathbf{T}^-$ to the target $\mathbf{z}_1$ along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_0)$ solution, which lies totally in curve $\mathbf{T}^-$, is expressed by:

$$T(\mathbf{z}, \mathbf{z}_1) = \left| \begin{array}{ll} y - \dfrac{A\sqrt{2(1+A)x_1}}{1+A}, & y \geqslant y_y; \\ \dfrac{y}{1+A}, & y \in [0, y_y]. \end{array} \right. \tag{3.29}$$

Trajectory of the $\mathbf{q}_+(t;\mathbf{z}_0)$ solution starting from $\mathbf{z}_0 = (0,0) = \mathbf{0}$ intersects the curve $\mathbf{T}^-$ in the point $\mathbf{z}_n \in \mathbf{T}^- \cap \mathbf{S}^+$ (see Figure 7), the co-ordinates of which are given by:

$$x_n = \frac{(1+A)x_1}{2}, \qquad y_n = \sqrt{\frac{x_1}{1-A^2}}. \tag{3.30}$$

A time taken for the transfer the object from $\mathbf{z}_0 = \mathbf{0}$ to $\mathbf{z}_n \in \mathbf{T}^-$ along the trajectory of the $\mathbf{q}_+(t;\mathbf{z}_0)$ and next along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_n)$ solution (that lies totally in $\mathbf{T}^-$) to the target $\mathbf{z}_1$ is expressed by:

$$T(\mathbf{z}_0, \mathbf{z}_n, \mathbf{z}_1) = 2\sqrt{\frac{x_1}{1-A^2}}. \tag{3.31}$$

Let us consider a transfer of the object from $\mathbf{z}_0 = \mathbf{0}$ to a state $\mathbf{z}_w = (0, y_w) \in \mathbf{B}^+$ in the following way: along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution to any point $\mathbf{z}_s \in \mathbf{S}^-$ and next from $\mathbf{z}_s$ to $\mathbf{z}_w = (0, y_w) \in \mathbf{B}^+$ along the trajectory of the $\mathbf{q}_+(t;\mathbf{z}_s)$ solution (see Figure 6). Let us define the co-ordinates of the state $\mathbf{z}_s$ as the functions by y_w. We have:

$$x_s = -\frac{y_w^2}{2}, \qquad y_s = -\frac{y_w}{\sqrt{2}}. \tag{3.32}$$

The transfer of the object from $\mathbf{z}_0 = \mathbf{0}$ along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_0)$ solution over the set $\mathbf{S}^-$ to $\mathbf{z}_s$ and from this point along the trajectory of $\mathbf{q}_+(t;\mathbf{z}_s)$ solution to $\mathbf{z}_w \in \mathbf{B}^+$ takes time given by the following formula:

$$T(\mathbf{0}, \mathbf{z}_s, \mathbf{z}_w) = y_w(1+\sqrt{2}). \tag{3.33}$$

The transfer of the object from $\mathbf{z}_w \in \mathbf{B}^+$ along the trajectory of the $\mathbf{q}_+(t;\mathbf{z}_w)$ solution to $\mathbf{z}_n \in \mathbf{T}^- \cap \mathbf{S}^+$ and from this point along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_n)$ solution to the target $\mathbf{z}_1$ takes time given by the following formula:

$$T(\mathbf{z}_w, \mathbf{z}_n, \mathbf{z}_1) = \frac{\sqrt{2}\sqrt{y_w^2 + 2(1-A)x_1}}{(1-A)\sqrt{1+A}} - \frac{y_w}{1-A}. \tag{3.34}$$

The co-ordinates of the state $\mathbf{z}_n \in \mathbf{T}^- \cap \mathbf{S}^+$ is given by:

$$x_n = \frac{2(1+A)x_1 - y_w^2}{4}, \qquad y_n = \sqrt{\frac{2(1-A^2)x_1 + (1+A)y_w^2}{2}}. \tag{3.35}$$

Time taken for the transfer of the object from $\mathbf{z}_0$ to $\mathbf{z}_s$ along the trajectory of $\mathbf{q}_-$ solution, from $\mathbf{z}_s$ to $\mathbf{z}_w$ along the trajectory of $\mathbf{q}_+$ solution, from $\mathbf{z}_w$ to $\mathbf{z}_n$ along the trajectory of $\mathbf{q}_+$ solution and finally from $\mathbf{z}_n$ to $\mathbf{z}_1$ along the trajectory of $\mathbf{q}_-$ solution is given as:

$$\begin{aligned} &T(\mathbf{0}, \mathbf{z}_s, \mathbf{z}_w, \mathbf{z}_n, \mathbf{z}_1) = T(\mathbf{0}.\mathbf{z}_s, \mathbf{z}_w) + T(\mathbf{z}_w, \mathbf{z}_n, \mathbf{z}_1) = \\ &\quad \alpha(\beta y_w + \sqrt{y_w^2 + \delta}), \quad y_w \in [0, y_y] \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} \alpha &= \frac{\sqrt{2}}{(1-A)\sqrt{1+A}}, \quad \beta = \frac{(\sqrt{2} - A - A\sqrt{2})\sqrt{1+A}}{\sqrt{2}}, \\ \delta &= 2(1-A)x_1. \end{aligned} \tag{3.37}$$

If only $\beta \geqslant 0$, i.e. if $A \leqslant 2 - \sqrt{2}$ then equation (3.36) takes its minimum for $y_w = 0$. This means that optimal transfer of the system from $\mathbf{z}_0 = \mathbf{0}$ to the target $\mathbf{z}_1$ should be executed along the trajectory $\mathbf{q}_+(t;\mathbf{z}_0)$ running over the $\mathbf{S}^+$ set to the point $\mathbf{z}_n \in \mathbf{T}^-$ and from $\mathbf{z}_n$ along the trajectory of the $\mathbf{q}_-(t;\mathbf{z}_n)$, i.e. along the curve $\mathbf{T}^-$. This completes the proof of *Thesis (b)*.

In order to prove *Thesis (a)* we will consider (3.36) under assumption $\beta < 0$, i.e. if $A \in (2-\sqrt{2}, 1)$.

The time defined by (3.36) depends on A, x_1 and y_w only. So, we will note it as $T(\mathbf{0}, \mathbf{z}_s, \mathbf{z}_w, \mathbf{z}_n, \mathbf{z}_1) = T(A, x_1, y_w)$ and its derivative towards y_w by

$$[T(A, x_1, y_w)]' = \frac{\delta T(A, x_1, y_w)}{\delta y_w}. \tag{3.38}$$

After calculation derivative (3.37) we get

$$[T(A, x_1, y_w)]' = \frac{\alpha}{\sqrt{y_w^2 + \delta}}(\beta\sqrt{y_w^2 + \delta} + y_w^2). \tag{3.39}$$

Calculating equation $[T(A, x_1, y_w)]' = 0$ we get that the function (3.36) reaches its extremum for

$$y_w + \bar{y}_w = \frac{(A + A\sqrt{2} - \sqrt{2})\sqrt{2x_1(1 + A)}}{\sqrt{A[(1 + A)(3 + 2\sqrt{2}) - 1]}} > 0. \tag{3.40}$$

Testing the sign of derivative (3.38) in a neighbourhood of $\bar{y}_w$ we state

$$T(\mathbf{0}, \mathbf{z}_s, \bar{\mathbf{z}}_w, \mathbf{z}_n, \mathbf{z}_1) = \min\{T(\mathbf{0}, \mathbf{z}_s, \mathbf{z}_w, \mathbf{z}_n, \mathbf{z}_1)\}.$$

The above completes the proof of the *Thesis (a)* and of the Lemma. ■

Setting (3.40) into (3.36) we get the minimum-time taken for the transfer the object to the target state $\mathbf{z}_1$. Thus

$$T(A, x_1, y_w)|_{\min} = T(A, x_1, \bar{y}_w) = \frac{\alpha}{\sqrt{\bar{y}_w^2 + \delta}}(\beta\sqrt{\bar{y}_w^2 + \delta} + \bar{y}_w^2) \tag{3.41}$$

where $\bar{y}_w$ is given by (3.40).

REMARK 3.7. If the motion resistance function satisfies (3.6, i), i.e. $A \in [0, 2 - \sqrt{2}]$ the optimal transfer of the object from $\mathbf{z}_0$ to the target $\mathbf{z}_1$ should be executed along the trajectory of $\mathbf{q}_+(t; \mathbf{z}_0)$ solution to the point $\mathbf{z}_n \in \mathbf{T}^-$ and from $\mathbf{z}_n$ along the trajectory of $\mathbf{q}_-(t; \mathbf{z}_n)$, i.e. along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$. This control process is realised with one switching operation in the state $\mathbf{z}_n \in \mathbf{T}^-$.

If the motion resistance function satisfies (3.6, ii), i.e. $A \in (2 - \sqrt{2}, 1)$ then the optimal transfer of the object from $\mathbf{z}_0$ to the target $\mathbf{z}_1$ should be executed along the trajectory of $\mathbf{q}_-(t; \mathbf{z}_0)$ throw the set $\mathbf{S}^-$ to the point $\mathbf{z}_s \in \mathbf{S}^-$, from $\mathbf{z}_s$ along the trajectory of $\mathbf{q}_+(t; \mathbf{z}_s)$ solution over the point $\mathbf{z}_n \in \mathbf{T}^-$ and from $\mathbf{z}_n$ along the trajectory of $\mathbf{q}_-(t; \mathbf{z}_n)$ i.e. along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$. This control process is realised with two switching operations executed in the point $\mathbf{z}_s$ and $\mathbf{z}_n$ one. ■

4. Non-Unique Time-Optimal Trajectories

Let us denote $\mathbf{T}_0^-$ the trajectory of such $\mathbf{q}_-(t; \mathbf{z}_0)$ solution that reaches the target $\mathbf{z}_1 = \mathbf{0}$ (see Figure 8). This trajectory has been already described by (2.3). So,

$$\mathbf{T}_0^- = \left\{(x, y) : x = -\frac{y^2}{2}, y \geqslant 0\right\}. \tag{4.1}$$

Obviously, $\mathbf{T}_0^- \subset \mathbf{S}^- \cup \{\mathbf{0}\}$. As previously by $\mathbf{z}_y$ we denote the point in which the switching curve $\mathbf{T}^-$ intersects semi-y-axis $\mathbf{B}^+$ (see Figure 6). Thus, $\mathbf{z}_y = (0, y_y) \in \mathbf{T}^- \cap \mathbf{B}^+, y_y > 0$.

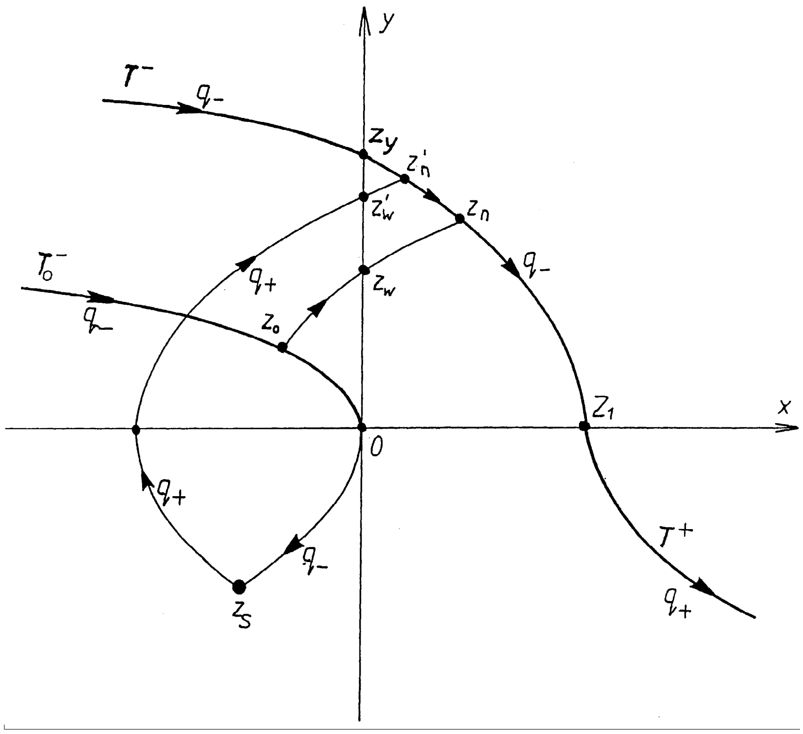

Figure 8. Non-unique trajectories

THEOREM 4.1. *Given a controlled object (1.1), (1.2). Let us assume that* $A \in (2-\sqrt{2}, 1)$. *There exists such a point* $\mathbf{z}_0 \in \mathbf{T}_0^- \backslash \{\mathbf{0}\}$ *from which there start two different bang-bang solutions the trajectories of which reach the target* $\mathbf{z}_1 = (x_1, 0), x_1 > 0$ *in the same minimum time* $t^* < \infty$.

Proof. (I). Trajectory of $\mathbf{q}_-(t; \mathbf{z}_0)$ starting from $\mathbf{z}_0 \in \mathbf{T}_0^- \backslash \{\mathbf{0}\}$ lies in the curve $\mathbf{T}_0^-$ and tends to reach the origin (0, 0) in a finite time. After leaving the origin it penetrates again into $\mathbf{S}^-$ set. Let in any point $\mathbf{z}_s \in \mathbf{S}^-$ there is executed switching operation. Thus, from $\mathbf{z}_s$ there starts the trajectory of $\mathbf{q}_+(t; \mathbf{z}_s)$ solution that running over $\mathbf{S}^-$ set intersects negative semi-x-axis and finally intersects the semi-y-axis $\mathbf{B}^+$ in the point $\mathbf{z}'_w = (0, y'_w) \in \mathbf{B}^+$. Let us assume that $y'_w \in [0, y_y]$ as it is shown in Figure 8.

The time taken for the transfer of the object from $\mathbf{z}_0$ along $\mathbf{T}_0^-$ to the origin (0, 0), next along the trajectory of $\mathbf{q}_-(t; \mathbf{0})$ to the point $\mathbf{z}_s \in \mathbf{S}^-$ and after executing the switching operation in $\mathbf{z}_s$ the transfer is continued along the trajectory of $\mathbf{q}_+(t; \mathbf{z}_s)$ solution over the point $\mathbf{z}'_w = (0, y'_w) \in \mathbf{B}^+$ to the point $\mathbf{z}'_n \in \mathbf{T}^-$ (i.e. the point of intersection with this part of the switching curve $\mathbf{T}^-$ that belongs to the set $\mathbf{S}^+$) and next along the trajectory of $\mathbf{q}_-(t; \mathbf{z}'_n)$, i.e. along the curve $\mathbf{T}^-$ to the target $\mathbf{z}_1$, is given by the following expression:

$$T(\mathbf{z}_0, \mathbf{0}, \mathbf{z}_s, \mathbf{z}'_w, \mathbf{z}'_n, \mathbf{z}_1) = \frac{\sqrt{4x_1(1-A)+2y'_w}}{(1-A)\sqrt{1+A}} - \frac{Ay'_w}{1-A} + \sqrt{2}y'_w + y_0. \tag{4.2}$$

Now, we are going to find such a value of y'_w which minimises the time $T(\mathbf{z}_0, \mathbf{0}, \mathbf{z}_s, \mathbf{z}'_w, \mathbf{z}'_n, \mathbf{z}_1)$. We must therefore find the solution of the following derivative:

$$\frac{\partial T(\mathbf{z}_0, \mathbf{0}, \mathbf{z}_s, \mathbf{z}'_w)}{\partial y'_w} = 0. \tag{4.3}$$

The solution of (4.3) is equivalent to the solution of the following equation:

$$d_4(y'_{w'})^4 + d_2(y'_w)^2 + d_0 = 0, \quad y'_w \in [0, y_y) \tag{4.4}$$

where:

$$\begin{aligned} d_4 &= A^2(A^2 - 4A - 4), \quad d_2 = 4A(-A^4 + 5A^3 - 10A - 4)x_1, \\ d_0 &= 4x_1^2(A^3 - 3A^2 - 2A + 2)^2 \end{aligned}$$

Solving (4.4) towards y'_w we get:

$$\begin{aligned} & y'_w = y'_w(y_0, x_1, A) = \\ & (A - 2 + \sqrt{2})\sqrt{\frac{2x_1(1+A)}{A(A-2+2\sqrt{2})}} = \sqrt{2(1+A)x_1\left(1 - \frac{2-\sqrt{2}}{A}\right)}. \end{aligned} \tag{4.5}$$

Going by assumption $A \in (2-\sqrt{2}, 1)$ and (3.28) simple estimation of (4.5) shows that $y'_w \in (0, y_y)$, what confirms presupposition $y'_w \in [0, y_y]$ taken when starting with the proving.

After setting (4.5) into (4.3) we get:

$$\begin{aligned} & T(\mathbf{z}_0, \mathbf{0}, \mathbf{z}_s, \mathbf{z}'_w(y_0, x_1, A), \mathbf{z}'_n, \mathbf{z}_1) = \\ & \quad T_{2\min}(y_0, x_1, A) = \sqrt{\frac{2x_1A(3A + 2\sqrt{2}A + 2)}{1+A}} + y_0 \end{aligned} \tag{4.6}$$

where $\mathbf{z}'_w = (0, y'_w(y_0, x_1, A))$.

Index "2" in (4.6) informs that the time-optimal transfer has been done with two switching operations (see Figure 8).

(II). Let us investigate the transfer of the object from $\mathbf{z}_0 \in \mathbf{T}_0^- \setminus \{\mathbf{0}\}$ along the trajectory of the $\mathbf{q}_+(t; \mathbf{z}_0)$ solution which after intersecting semi-y-axis $\mathbf{B}^+$ in the point $\mathbf{z}_w = (0, y_w) \in \mathbf{B}^+$. Let us assume that $y_w \in [0, y_y]$ as it is shown in Figure 8. This assumption does that trajectory penetrates into $\mathbf{S}^+$ set and next intersects switching curve $\mathbf{T}^-$ in the point $\mathbf{z}_n \in \mathbf{T}^- \cap \mathbf{S}^-$. This plays an essential role in computing a time of transfer the system to the target $\mathbf{z}_1$. From $\mathbf{z}_n \in \mathbf{T}^-$ the object is brought to the target $\mathbf{z}_1$ along the trajectory of $\mathbf{q}_-(t;\mathbf{z}_n)$ solution lying totally in $\mathbf{T}^-$ switching curve. The time taken for the above mentioned transfer is given by the formula:

$$T(\mathbf{z}_0, \mathbf{z}_w, \mathbf{z}_n, \mathbf{z}_1) =$$
$$T_1(y_0, x_1, A) = \frac{2\sqrt{x_1(1-A)+y_0^2}}{(1-A)\sqrt{1+A}} - \frac{\sqrt{2}Ay_0}{1-A} - y_0. \tag{4.7}$$

Index “1” in (4.7) informs that the time-optimal transfer has been done with one switching operation only (see Figure 8).

Now, we are going to compare equations (4.6) with (4.7), i.e.

$$T_{2\,\mathrm{min}}(y_0, x_1, A) = T_1(y_0, x_1, A). \tag{4.8}$$

Expression (4.8) is equivalent to the following equation:

$$r_2 y_0^2 + r_1 y_0 + r_0 = 0 \tag{4.9}$$

where

$$\begin{aligned} r_2 &= 2(3-2\sqrt{2})A(1-A)(1+\sqrt{2}-A), \\ r_1 &= 2(2-\sqrt{2})A(1-A)(2+\sqrt{2}-A)\sqrt{2x_1A(3+2\sqrt{2})(1+A)(-2+2\sqrt{2}+A)} \\ r_0 &= -2(3+2\sqrt{2})x_1(1-A^2)(A-2+\sqrt{2})^2. \end{aligned}$$

Positive root of (4.9) is given by:

$$y_0 = y_0(x_1, A) =$$
$$\sqrt{\frac{x_1(1+A)}{2A}}\left(\frac{4\sqrt[4]{2}-(2-2A+\sqrt{2}A)\sqrt{2+2\sqrt{2}+3A+2\sqrt{2}A}}{-2+2\sqrt{2}-3A+2\sqrt{2}A}\right). \tag{4.10}$$

This completes the proof. ∎

The co-ordinate $x_0 = x_0(x_1, A)$ of the state $\mathbf{z}_0 = (y_0(x_1,A), x_0(x_1,A)) \in \mathbf{T}_0^-$ from which there start two trajectories of two, non-unique time-optimal solutions we get after setting into (4.1) expression (4.10). We get:

$$x_0 = x_0(x_1, A) =$$
$$-\frac{1}{2}\left(\frac{x_1(1+A)}{2A}\left(\frac{4\sqrt[4]{2}-(2-2A+\sqrt{2}A)\sqrt{2+2\sqrt{2}+3A+2\sqrt{2}A}}{-2+2\sqrt{2}-3A+2\sqrt{2}A}\right)^2\right) < 0. \tag{4.11}$$

If $A \in (2-\sqrt{2},1)$ then repeating the same way of computing as that done in the proof of Theorem 4.1 we state that there exists a subset of the states $\mathbf{z}_0 = (x_0, y_0) \in \mathbf{S}^-$ from which there start the trajectories of non-unique time-optimal solutions. These co-ordinates x_0, y_0 may be found from solution of 4-the degree algebraic equation. Unfortunately, those co-ordinates cannot be defined in an open form such as that in the proof of Theorem 4.1, equations (4.10), (4.11). They may be calculated in numerical way only.

From the point of view of time-optimal closed-loop system synthesis knowing the values of these co-ordinates does not play an essential role. More important is knowledge, that in the state plane there does exist the state from which there start the non-unique time-optimal trajectories. The *singular phenomenon* of existence of non-unique time-optimal trajectories will be a basic point in the next paragraph where there will be given some proposals as to practical applications.

5. Concluding Remarks

Knowledge of time-optimal solution plays an essential role in practical applications. Usually, there is created a closed-loop system which attributes to each of the state a time optimal value of the control function u. Thus, the open controlled system $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{u}), \mathbf{z} \in R^n, \mathbf{u} \in \mathbf{U} \subset R^m$ is replaced by a feedback system $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{v}(\mathbf{z}))$, where control function $\mathbf{v} : R^n \to \mathbf{U}$. This way of feedback system synthesis is based on so called *Method of Regular Synthesis* [2], [6] that establishes that: (a) each time-optimal solution of the open controlled object $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{u})$ is a standard (Caratheodory) solution of the mentioned above closed-loop system $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{v}(\mathbf{z}))$, (b) each standard solution of that closed-loop system is a time-optimal solution of that open, controlled object. It should be emphasized that the above concept application requires the uniqueness of time-optimal solution.

For the desirability of implementing the above closed-loop time-optimal system the following reasons may be given: (1) *There is no need to compute the optimal control for every new initial state separately*. (2) *The controller acting upon* $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{v}(\mathbf{z}))$ *is sensitive to instantaneous perturbations, i.e. if at any instant of the process the system is deviated from its optimal trajectory, the remaining portion of the process will again lead to the desired final state (target) and will be optimal with respect to this new initial state*.

It should be strongly emphasized that the above concept of feedback system application requires the uniqueness of time-optimal solution.

In the case investigated in this paper the unique time-optimal solution exists merely if $A \leqslant A_b = 2-\sqrt{2}$. Then the closed-loop system may be synthesised in standard way as shown previously in the text. Instead, if $A \in (2-\sqrt{2},1)$ then the switching operation of the control function should be executed on the curve which cannot be formed by trajectory of any one solution of the system. Moreover, there exists a set of the states from which there start the trajectories of non-unique time-optimal solutions. In this case there is practically impossible to create the closed-loop system which generates the time-optimal control function depending on the states of the investigated controlled object.

The feedback system of the type $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{v}(\mathbf{z}))$ is very attractive from technical point of view because of the properties shown above. If a designer of the controlled system may accomplish such a selection of the elements composing the controlled system that the relation $A \leqslant A_b$ is satisfied then the feedback system takes a standard form with switching curve formed in typical way. Instead, if such a way in treatment of synthesis process appears impossible, then a creation of the sub-optimal feedback system acting with the use of standard switching curve becomes the unique one way of suitable feedback system formation.

References

1. Binding, P. (1979), Differential equations $\dot{x} = f \circ x$. *J. Diff. Equations* **31**, 183–189.
2. Brunovsky, P. (1974), The closed-loop time-optimal control. *SIAM J. Control* **12**, 624–634.
3. Filippov, A.F. (1964), Differential equations with discontinuous right-hand side. *Trans. Amer. Math. Soc. Ser. 2* **42**, 199–231.
4. Hajek, O. (1979), Discontinues differential equations. *J. Diff. Equations* **32**, 149–185.
5. Hejmo, W. and Kloch J. (1984), On the time-optimal problem of positional control with discontinuous resistance of motion. *RAIRO-System Analysis and Control* **18**, 329–341.
6. Hejmo, W. and Kloch J. (1989a), On the perturbations in a time-optimal closed-loop system. *Annales Polonici Mathematici* **50**, 37–52.
7. Hejmo, W. and Kloch J. (1989b), Switching Curve Deformation in a Time-Optimal Closed-Loop System. *IEEE. Tr. on Ac.* **34**, 654–656.
8. Sussman, H.J. (1980), The structure of time-optimal trajectories for single-input systems in the plane: C non-singular case. *J. Diff. Equations* **25**, 433–465.
9. Utkin, V.I. (1971), Equations of Slipping Regime in Discontinuous Systems, I, II, *Automation and Remote Control* **32**, 1897–1907, and (1972) **32**, 211–219.
10. Young, D.S. (1977), Time-optimal feedback control. *J. Optimal Theory Applications* **21**, 71–82.